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Critical phenomena in globally coupled excitable elements

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Critical phenomena in globally coupled excitable elements are studied by focusing on a saddle-node bifurcation at the collective level. Critical exponents that characterize divergent fluctuations of interspike intervals near the bifurcation are calculated theoretically. The calculated values appear to be in good agreement with those determined by numerical experiments. The relevance of our results to jamming transitions is also mentioned.

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The understanding of cooperative phenomena in nonequilibrium systems is one of the most important problems in physics. In contrast to equilibrium systems, their nature essentially depends on the dynamical properties of the systems. This makes it difficult to perform a systematic study of the problem. Thus, it is necessary to investigate the typical cooperative phenomena in nonequilibrium systems.

Recently, critical behaviors in neural networks have been observed in experiments [1] and mathematical models [2,3], which are typical examples of coupled excitable elements. In general, such critical behaviors are classified into several groups on the basis of the exponents of divergences. The classification enables the realization of a universality class for critical phenomena in coupled excitable elements. However, the broad distribution of the phenomena makes it difficult to elucidate the mechanism of the criticality. When we consider the role of the mean-field Ising model in theories of equilibrium statistical mechanics, it becomes apparent that it is necessary to develop an elegant method for theoretical analysis of a minimal model describing critical phenomena in coupled excitable elements.

Thus, with this aim in mind, we analyze a previously proposed simple model for globally coupled excitable elements in this Rapid Communication [4]. In particular, we focus on a divergent behavior with respect to parameter change around a saddle-node bifurcation because the excitability of this model is related to the bifurcation. It should be noted that such transition properties have been studied for different excitable systems [2]. The main achievement of this Rapid Communication is a theoretical derivation of the critical exponents that characterize the singular behavior near a saddle-node bifurcation.

Model. The excitable nature of a system is characterized by the existence of spikes in a time series. Mathematically speaking, spikes can be described by trajectories near a homoclinic orbit in a differential equation. As a simple example, let us consider an ordinary differential equation $\partial_t\phi=\omega-h\sin\phi$ for a phase variable $\phi\in[0,2\pi]$, in which there exists the homoclinic orbit when $\omega=h$. Then, when h is slightly larger than ω, there are stable and unstable fixed points satisfying $\phi=\sin^{-1}(\omega/h)$ and a small perturbation for the stable fixed point ϕ_* yields one spike. On the other hand, when h is slightly less than ω, the system shows an array of spikes with a long interspike interval. The qualitative change in the trajectories is an example of saddle-node bifurcation. By using this simple dynamics as a phase model of excitable elements, we study globally coupled excitable elements $\{\phi_i\}_{i=1}^N$ under the influence of noise [4],

$$\partial_t\phi_i=\omega-h\sin\phi_i-\frac{K}{N}\sum_{j=1}^{N}\sin(\phi_i-\phi_j)+\xi_i, \tag{1}$$

where $\xi_i(t)$ is Gaussian white noise that satisfies the relation $\langle\xi_i(t)\xi_j(t')\rangle=2T\delta_{i,j}\delta(t-t')$. Without loss of generality we can assume $K=1$, and we restrict our investigations to the case $\omega=1$. The control parameters are h and T. All numerical results in this study have been calculated by employing the forward discretization method with a time step $\delta t=0.05$, in which the extent of the error is estimated as $O((\delta t)^{2/3})$ in one integration step. Note that the midpoint discretization method with the error term of $O((\delta t)^2)$ provides the same result as that reported in this paper when δt is sufficiently small.

The collective behavior of this system is described by the time evolution of a complex amplitude

$$Z\equiv\frac{1}{N}\sum_{j=1}^{N}e^{i\phi_j}. \tag{2}$$

In particular, for the expression $Z=X+iY$, where X and Y are real numbers, the expectation values of the angular momentum $L\equiv X(\partial_tY)-(\partial_tX)Y$ are used to distinguish the oscillatory states ($\langle L\rangle\neq0$) from the stationary states ($\langle L\rangle=0$) [5]. In Fig. 1, we show an approximate phase diagram in the form of a curve that satisfies the condition $0.097<\langle L\rangle<0.107$ in the parameter space (T,h). A similar phase diagram was obtained in Ref. [4] by investigating the behavior of the time-dependent phase distribution function. Here, the curve starting from $(T,h)\simeq(0,1)$ is related to a saddle-node bifurcation, while the curve from $(T,h)\simeq(0.5,0)$ is related to a Hopf bifurcation. It should be noted that a complicated bifurcation diagram appears near $T=0.1$, which originates from the Takens-Bogdanov type bifurcation [6].

Preliminary. In this study, we focus on systems near a saddle-node bifurcation. First, we fix $T=0.05$ and change h across the bifurcation from below. Here, the amplitude of

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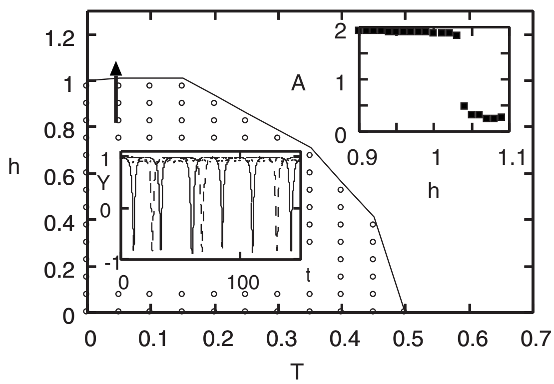

FIG. 1. Phase diagram. $\langle L\rangle$ on the displayed curve satisfies $0.097<\langle L\rangle<0.107$. Inset: (Right-hand side) Amplitude of oscillation $A\equiv\langle[\max_t X(t)-\min_t X(t)]\rangle$ as a function of h. (Left-hand side) Typical samples of the time evolution of Y for $h=1.0$ (solid line), $h=1.02$ (dashed line), and $h=1.1$ (dotted line). Here, $T=0.05$ and $N=100$. The arrow represents the direction of the parameter change. $\langle L\rangle>0.1$ in the states where the symbols are placed.

oscillation $A\equiv\langle[\max_t X(t)-\min_t X(t)]\rangle$ changes discontinuously at the bifurcation. (See inset of Fig. 1.) The discontinuous change in the amplitude is in sharp contrast to a supercritical Hopf bifurcation at a collective level, where the amplitude of oscillation changes continuously at the bifurcation [7]. It should be noted that in a manner similar to that of the critical phenomenon in equilibrium systems, the continuous transition leads to a critical divergence of amplitude fluctuation [8]. (See also Refs. [9,10] as reviews.) On the contrary to such usual cases, the discontinuous nature of the transition is not indicative of the appearance of critical phenomena.

Nevertheless, based on the fact that a typical time scale diverges at a saddle-node bifurcation, we take into account the fluctuation of interspike intervals. Explicitly, by using the phase of collective oscillation

$$\theta\equiv\arg(Z), \tag{3}$$

we define the interspike interval $\hat{I}$ as the minimum time interval $[t,t+\hat{I}]$ over which the time integration of $\partial_t\theta$ is equal to 2π for a time t satisfying $\theta(t)=-\pi/2$. As the most primitive statistical quantities of $\hat{I}$, we measured its average and fluctuation intensity defined by

$$I_*(h,N)\equiv\langle\hat{I}\rangle, \tag{4}$$

$$\chi(h,N)\equiv N(\langle\hat{I}^2\rangle-\langle\hat{I}\rangle^2). \tag{5}$$

In order to determine the divergent behaviors near the bifurcation in the thermodynamic limit, we performed finite-size scaling analysis by using systems with $N=10$, 100, and 1000. For each system, the values of $I_*(h,N)$ and $\chi(h,N)$ were calculated for several values of h. Then, we expect that there exists a correlation size for a given $(h_c-h)/h_c$ and that its size diverges as $[(h_c-h)/h_c]^{-\nu}$. On the basis of this expectation, we assume the scaling relations

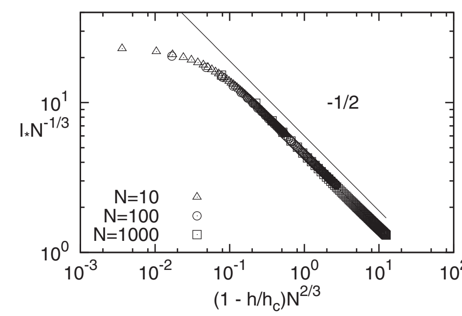

FIG. 2. $I_*N^{-1/3}$ as a function of $(1-h/h_c)N^{2/3}$. Here, $h_c=1.0283$. The guide line represents a power-law function with exponent $-1/2$.

$$I_*(h,N)\simeq N^{\zeta/\nu}F_I\left(\frac{h_c-h}{h_c}N^{1/\nu}\right), \tag{6}$$

$$\chi(h,N)\simeq N^{\lambda/\nu}F_\chi\left(\frac{h_c-h}{h_c}N^{1/\nu}\right), \tag{7}$$

where the exponents ν, ζ, and λ and the critical value h_c are determined so that the scaling relations are valid. We also assume that a distribution function of $\hat{I}$ is expressed as a function of $\hat{I}N^{\zeta/\nu}$ when $h=h_c$. By applying this assumption to $\chi(h,N)$ in (5), we find a relation $\lambda/\nu=2\zeta/\nu+1$, which yields

$$\nu=\lambda-2\zeta. \tag{8}$$

Moreover, since I_* and χ are independent of N in the regime $(h_c-h)N^{1/\nu}\gg 1$, the asymptotic behaviors can be derived as $F_I(x)\simeq x^{-\zeta}$ and $F_\chi(x)\simeq x^{-\lambda}$. With the consideration of these conditions, we determine the values $h_c=1.0283$, $\nu=3/2$, $\zeta=1/2$, and $\lambda=5/2$, for which the excellent collapses to universal curves are found, as displayed in Figs. 2 and 3.

Theory. We now present a theory for the results $\zeta=1/2$ and $\lambda=5/2$. [ν is then determined from (8).] In the argument below, we assume that $\epsilon\equiv h_c-h$ is a sufficiently small positive constant and consider the asymptotic limit $N\to\infty$ for the assumed value of ϵ.

We first notice that for a sufficiently small value of T, the excitable elements are almost in synchronization. Thus,

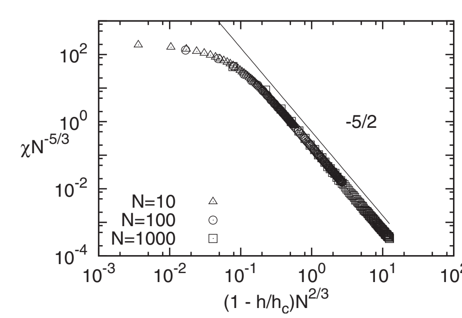

FIG. 3. $\chi N^{-3/2}$ as a function of $(1-h/h_c)N^{2/3}$. Here, $h_c=1.0283$. The guide line represents a power-law function with exponent $-5/2$.

when setting $\phi_i=\theta+\delta\phi_i$, we assume that $|\delta\phi_i| \ll 1$. From this assumption and the definition of θ given in (3), we can derive the equation

$$\partial_t \theta = \omega - h \sin\theta + \eta, \tag{9}$$

with $\langle \eta(t)\eta(t')\rangle = 2(T/N)\delta(t-t')$, where we have ignored the contribution of $O(\Sigma_{i=1}^N (\delta\phi_i)^2/N)$ to the time evolution of θ. Within this approximation, h_c is determined as $h_c=\omega$. Although the equation we analyze has become quite simple, the calculation of the critical exponents is still nontrivial. In the recent study [11], the divergent fluctuation of the exit time from a region near $\theta \simeq \pi/2$ has been derived for a model which is expected to be essentially the same as (9). Since the calculation requires some special knowledge on the stochastic process [12], we present a different method by which the values of the exponents can be determined. When we consider high-dimensional systems, our proposing method might be more useful than the previous one.

The basic idea of our analysis is to consider a distribution function of the time averaged frequency $\hat{\Omega}$ over a time interval $\Delta t = MI_*$, where M is a large number independent of ϵ. (Note that Δt depends on ϵ.) For the explicit expression

$$\hat{\Omega} = \frac{1}{\Delta t}\int_0^{\Delta t} dt(\partial_t\theta), \tag{10}$$

we can expect a large deviation property given as

$$P(\hat{\Omega}=\Omega) \simeq e^{-MNG(\Omega)/T}, \tag{11}$$

where the rate function $G(\Omega)$ takes a minimum value zero when $\Omega=\Omega_*$. Then, it can be shown that I_* in (4) is equal to $2\pi/\Omega_*$.

We now estimate the rate function $G(\Omega)$. Let $[\theta]$ be a trajectory $(\theta(t))_{t=0}^{\Delta t}$, and $\theta(0)$ is fixed as an arbitrary value. The probability density of the trajectory is then expressed by

$$\mathcal{P}([\theta]) = \frac{1}{Z} e^{-(N/4T)\int_0^{\Delta t} dt\{[\partial_t\theta - f(\theta)]^2 + (2T/N)f'(\theta)\}}, \tag{12}$$

where $f(\theta)=\omega - h\sin\theta$, the prime represents the derivative with respect to θ, and Z is a normalization factor. The last term corresponds to a Jacobian term associated with the transformation from a noise sequence $(\eta(t))_{t=0}^{\Delta t}$ to the trajectory $[\theta]$, where the midpoint discretization is employed in order to ensure standard calculus. By formally expressing $P(\hat{\Omega}=\Omega)$ as

$$P(\hat{\Omega}=\Omega) = \int \mathcal{D}[\theta]\mathcal{P}([\theta])\,\delta\!\left(\Omega - \frac{1}{\Delta t}\int_0^{\Delta t} dt(\partial_t\theta)\right), \tag{13}$$

we consider the trajectory whose weight becomes most dominant in the limit $N\to\infty$. Note that the Jacobian term is irrelevant due to its $1/N$ contribution in this analysis. The trajectory, which is denoted by θ_Ω^s, is a periodic solution with period $2\pi/\Omega$ of the variational equation $\partial_t^2\theta(t) = -\partial_\theta U(\theta)/2$, where $U(\theta) = -f(\theta)^2$. The solution $\theta_\Omega^s(t)$ is obtained from the energy conservation equation, which leads to

$$\partial_t \theta_\Omega^s = \sqrt{E(\Omega) - U(\theta_\Omega^s)}, \tag{14}$$

where the parameter $E(\Omega)$ is determined as

$$\frac{2\pi}{\Omega} = \int_0^{2\pi} \frac{d\theta}{\sqrt{E(\Omega)-U(\theta)}}. \tag{15}$$

Since θ_Ω^s contributes to $P(\hat{\Omega}=\Omega)$ much more than other $2\pi/\Omega$-periodic trajectories, it is reasonable to expect that $P(\hat{\Omega}=\Omega) \simeq \mathcal{P}([\theta_\Omega^s])$. The substitution of (14) into (12) yields

$$G(\Omega) = \frac{I_*\Omega}{8\pi}\int_0^{2\pi} d\theta \frac{[\sqrt{E(\Omega)-U(\theta)} - \sqrt{-U(\theta)}]^2}{\sqrt{E(\Omega)-U(\theta)}}. \tag{16}$$

It can be observed that $dG(\Omega)/d\Omega|_{\Omega_*}=0$ and $G(\Omega_*)=0$ when Ω_* satisfies the condition $E(\Omega_*)=0$. Therefore, the rate function $G(\Omega)$ takes a quadratic form

$$G(\Omega) = B(\epsilon)\Omega_*^{-4}(\Omega-\Omega_*)^2 \tag{17}$$

when Ω is close to Ω_*, where $B(\epsilon)$ is calculated as

$$B(\epsilon) = \frac{8\sqrt{2}\pi}{3}\epsilon^{5/2} + O(\epsilon^{7/2}). \tag{18}$$

Furthermore, by considering (15) with $E=0$, we obtain

$$\Omega_* = \sqrt{2-\epsilon}\,\epsilon^{1/2}. \tag{19}$$

Now, we consider the average of $\hat{I}$ during the time interval Δt, which is denoted by $\hat{J}$. It can be easily confirmed that $\hat{J}=2\pi/\hat{\Omega}$. Then, by the transformation of the variable in (11) and (17), we derive

$$P(\hat{J}=J) \simeq e^{-MNB(\epsilon)(J-2\pi/\Omega_*)^2/(4\pi^2 T)}. \tag{20}$$

By substituting (18) and (19) into (20), we find that $\langle \hat{J}\rangle \simeq \epsilon^{-1/2}$ and $\langle(\hat{J}-\langle\hat{J}\rangle)^2\rangle \simeq \epsilon^{-5/2}$. Since these ϵ dependences should be equal to those of I_* and χ, we arrive at the theoretical results $\zeta=1/2$ and $\lambda=5/2$. These values coincide perfectly with the numerical values.

Furthermore, our analysis yields a formula for the phase diffusion constant D, which is expressed by $D \equiv \Delta t\langle(\hat{\Omega}-\Omega_*)^2\rangle/2$ because $\hat{\Omega}=[\theta(\Delta t)-\theta(0)]/\Delta t$. Indeed, from (11), we obtain

$$D = \frac{T}{2N}\frac{I_*}{G''(\Omega_*)}, \tag{21}$$

which leads to the power-law behavior $D = (3T/32\sqrt{2}\pi N)\epsilon^{-1} + O(\epsilon^0)$. Here, with the crossover relation $\epsilon \simeq (N/T)^{-2/3}$, we conjecture $D \simeq (T/N)(N/T)^{2/3}$ at $\epsilon=0$, which was reported in Ref. [13].

Concluding remarks. We have studied a simple model that exhibits critical behavior near a saddle-node bifurcation. We have calculated the power-law divergence $\chi \simeq \epsilon^{-5/2}$ for the model of coupled excitable elements. We expect that such a power law will be observed in real experimental systems. With regard to the connection to realistic systems, we will analyze complicated systems such as those with a tactical network or integrated-fire dynamics by extending our theory.

The analysis of finite-dimensional systems is the next theoretical problem, because the theory we presented might be applied to more general cases if excitable elements are almost synchronous near a saddle-node bifurcation. As usual in equilibrium, we wish to determine the upper-critical dimension above which the values of the exponents are the same as those in the globally coupled model. Then, we intend to develop a systematic method to take into account non-Gaussian fluctuations.

Before ending this paper, let us recall that the amplitude of oscillation exhibits a discontinuous transition at the saddle-node bifurcation. Here, it should be noted that the coexistence of critical fluctuations with a discontinuous transition is one of the remarkable features of jamming transitions [14]. This is not an accidental coincidence and can be explained in the following manner.

A standard characterization of the critical nature near a jamming transition is based on the nonlinear susceptibility $\chi_4(t)$, which quantifies the fluctuations of unlocking events during a time interval t [15]. Among the several theories for $\chi_4(t)$ [16–18], one theory states that the divergence of $\chi_4(t)$ originates from the critical fluctuations of the time when an unlocking event occurs [18]. By employing the method in Ref. [18], we can discuss the divergent behavior of amplitude fluctuations in the present problem. Moreover, it has been recently shown that dynamical behaviors of the k-core percolation in a random graph exhibit a saddle-node bifurcation at the percolation point [19]. Since it has been known that the k-core percolation is related to a kinetically constrained model and a random-field Ising model [20–24], the present work might be useful for theoretical analysis of such systems.

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